MTH 301 Quiz Solutions

1. Find up to isomorphism:

   (a) All abelain groups of order 16.

   (b) Two non-abelian groups of order 16, that are direct products.

   (c) Two groups of order 16 that are non-trivial semi-direct products.

   Solution. (a) Using the Classification Theorem for finitely-generated abelian groups, we see that there are 5 abelian groups of order 16 up to isomorphism, and they are

   (i) $\mathbb{Z}_{16}$

   (ii) $\mathbb{Z}_8 \times \mathbb{Z}_2$

   (iii) $\mathbb{Z}_4 \times \mathbb{Z}_4$

   (iv) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$

   (v) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

   These groups are non-isomorphic for the following reasons:

   - The integer $16 = 2^4$ factors precisely in five different ways (as listed above), and
   - $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \iff \gcd(m, n) = 1$.

   (b) Since D_8 and Q_8 are non-abelian and non-isomorphic groups of order 8, the direct products $D_8 \times \mathbb{Z}_2$ and $Q_8 \times \mathbb{Z}_2$ are non-abelian groups of order 16.

   (c) We know from class that the metacyclic group $D_{16} \cong \mathbb{Z}_2 \ltimes_{-1} \mathbb{Z}_8$ is a non-trivial semi-direct product of order 16.

   Since $\phi(4) = 2$, we have $\gcd(4, \phi(4)) > 1$, and consequently, there exists a non-trivial semi-direct product of the form $\mathbb{Z}_4 \ltimes \mathbb{Z}_4$. An explicit description of the structure of this group is left as an exercise.

2. Show that a group of order 36 is non-simple.

   (c) Since $36 = 2^2 3^2$, the First and Third Sylow's Theorems imply that either $n_3 = 1$ or $n_3 = 4$. If $n_3 = 1$, then there is a unique subgroup H

of order 9, and so the Second Sylow Theorem would imply that $H \trianglelefteq G$, showing that G is non-simple.

Suppose that $n_3 = 4$. We give two different methods for establishing the result.

Method 1: Let N and K be two distinct Sylow 3-subgroups. Then $N \cap K$ has to be a subgroup of order 3, for if $N \cap K = \{1\}$, then $|NK| = 81$, which is impossible. Furthermore, since

$$[N : N \cap K] = [K : N \cap K] = 3,$$

we have that

$$N \cap K \trianglelefteq N \text{ and } N \cap K \trianglelefteq K.$$

Consider the normalizer $N_G(N \cap K)$ of $N \cap K$ in G. Note that as $N, K \leq N_G(N \cap K)$ (Why?), we have that

$$NK \subset \langle N \cup K \rangle \leq N_G(N \cap K)$$

But since $|NK| = 27$ and $|N_G(N \cap K)| \mid 36$, we have that $|N_G(H \cap K)| = 36$. Therefore, $N_G(H \cap K) = G$, so $N \cap K \trianglelefteq G$, confirming the non-simplicity of G.

Method 2: Consider the set $Syl_3(G)$ of all Sylow 3-subgroups of G and the action $G \curvearrowright^c Syl_3(G)$. This action induces a permutation representation

$$\psi_c : G \to S_4$$

If $\operatorname{Ker} \psi_c$ is trivial, then $G \hookrightarrow S_4$, which is impossible as $|G| \nmid 24$. Hence, $\operatorname{Ker} \psi_c$ is a nontrivial normal subgroup of G, which shows that G is non-simple.

3. Without using the Feit-Thompson Theorem, show that a group of order pq, where p and q are distinct primes, is solvable.

Solution. Let G be a group of order pq, where p and q are distinct primes. Then by the First Sylow Theorem, G has a subgroup H of order p and a subgroup K of order q, which are both abelian. Without loss of generality, we may assume that $p < q$. Since $[G : K] = p$, we have that $K \trianglelefteq G$, and we obtain a normal series

$$1 \trianglelefteq K \trianglelefteq HK = G,$$

where both the factor groups are abelian. This shows that G is solvable.